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CHAPTER 5

MOTION OF EXTENDED BODIES IN METRIC THEORIES OF GRAVITATION

27. Determination of the Tensor of Passive Gravitational Mass

Among possible theories of gravitation metric theories occupy a special place, i.e., theories of gravitation according to which the action of the gravitational field on matter is realized in terms of a metric tensor of Riemannian space–time. The unified description in these theories of the motion of matter in a gravitational field makes it possible, apart from the details of one gravitational theory or another, to compute the motion of matter simultaneously for an entire class of metric theories of gravitation. It was just with this purpose Will and Nordtvedt [11] developed the parametrized post-Newtonian formalism which for particular values of the parameters coincide with the post-Newtonian limit of any metric theory of gravitation. Therefore, this formalism is rather widely used not only for the calculations of various experiments but also for the analysis of various general questions.

One of these is the question of the relation between the inertial and gravitational masses of an extended body in various metric theories of gravitation and the effect of this relation on the character of the motion of the center of mass of the body. Investigation of this question has been the focus of attention of a number of authors. In particular, Will [9], having in mind subsequent application of the results of his computations to the system including the sun and one of its planets, showed that the post-Newtonian equations of motion of the center of mass of an extended body (the planet) in the gravitational field of a point body at rest (the sun) have the form

$$Ma^{\alpha}=-\frac{M_0}{R^2}f^{\alpha}, \tag{27.1}$$

where M is the mass of the extended body, M_0 is the active gravitational mass of the point body at rest, a^{α} are the components of the acceleration of the center of mass of the extended body, and R is the distance between the point body and the center of mass of the extended body.

The following expression was obtained for the vector f^{α} in this case:

$$f^{\alpha}=M\{n^{\alpha}[1-(4\beta-\alpha_1-\gamma-3-\xi_1+\alpha_2)\Omega]+(\alpha_2-\xi_1+\xi_2)\Omega^{\alpha\beta}n_{\beta}\},$$

where $n^{\alpha} = R^{\alpha}/R$, and Ω and $\Omega^{\alpha\beta}$ are the post-Newtonian corrections defined by relations (18.3).

Will [9] defined the tensor of passive gravitational mass in correspondence with the equality

$$f^{\alpha}=-m_p^{\alpha\beta}n_{\beta}.$$

Because of this definition, he arrived at the conclusion that this tensor has the form

$$\frac{m_p^{\alpha\beta}}{M}=-\gamma^{\alpha\beta}[1-(4\beta-\alpha_1-\gamma-3-\xi_1+\alpha_2)\Omega]-(\alpha_2-\xi_1+\xi_2)\Omega^{\alpha\beta}. \tag{27.2}$$

On the basis of this expression and data obtained in the laser ranging of the moon, the authors of the works [8, 12] arrived at the conclusion that in the post-Newtonian approximation the passive gravitational mass of an extended body is equal to its inertial mass, and hence the center of mass of an extended body moves along geodesics of Riemannian space–time.

However, these conclusions are incorrect, since in obtaining formulas (27.1) and (27.2) Will assumed that the velocity of the extended body about the sun ($v \sim 10^{-4}$ c) was equal to zero. Consideration of the motion of the extended body, as we shall show below, in this determination of the tensor of the passive gravitational mass leads to the somewhat different formula

$$\frac{m_p^{\alpha\beta}}{M} = -\gamma^{\alpha\beta}[1-(2\beta+\gamma)v^2-(4\beta-\alpha_1-\gamma-3-\xi_1+\alpha_2)\Omega]-2(\gamma+1)v^\alpha v^\beta-(\alpha_2-\xi_1+\xi_2)\Omega^{\alpha\beta}.$$

In this case data obtained in the laser ranging of the moon with consideration of other experiments provide the basis for the assertion that in the post-Newtonian approximation the passive gravitational mass of an extended body is not equal to its inertial mass.

In this connection it should be noted that the equality or nonequality of the passive gravitational mass of an extended body so introduced to its inertial mass cannot serve as an indication of how the center of mass of the extended body moves: along a geodesic of space–time or not. As we shall see below, equality of these masses guarantees only the coincidence of the post-Newtonian equations of motion of the center of mass of the extended body with the corresponding equations of Newton's theory which in no measure is a condition that they coincide with the equations of geodesics.

In a subsequent work [10] Will did not make the assumption that the extended body was equal to zero, although he wrote the equations of the center of mass of the extended body contained in the binary system in the form

$$Ma^\alpha = m_p^{\alpha\beta}\partial_\beta \mathcal{U} + Ma_N^\alpha, \tag{27.3}$$

where the tensor $m_p^{\alpha\beta}$, depending on the characteristics of the extended body, he defined, as previously, by the expression (27.2). All the remaining terms on the right side of the equations of motion he called N-body accelerations and included them in the term a_N^α. However, this separation is arbitrary, since the quantity a_N^α, as is easily seen [see formulas (6.42), (6.44), (6.50)-(6.52) of the work [10]), contains terms of the same structure as the first term on the right side of (27.3):

$$a_N^\alpha = L_{(1)}^{\alpha\beta}\partial_\beta \mathcal{U} + q^\alpha.$$

Since the tensor $L_{(1)}^{\alpha\beta}$ depends on the characteristics of the first body (in particular, on the square of its velocity), it can be included in the expression for $m_p^{\alpha\beta}$ on the right side of the equations of motion (27.3).

Thus, the partition of expression (27.3) proposed by Will is arbitrary and is not in any way justified except by the desire to achieve at any price formal equality in the post-Newtonian approximation of the passive gravitational mass and the inertial mass of an extended body in the general theory of relativity. Now in Einstein's theory, and this is a general situation, there are no conservation laws of matter and gravitational field taken together. A special consequence of this situation, as shown in the first chapter of the present work, is the assertion that the inertial mass defined in the general theory of relativity has no physical meaning. Therefore, in Einstein's theory its comparison with the passive gravitational mass is physically meaningless.

However, in metric theories of gravitation possessing conservation laws of matter and gravitational field taken together, the concept of inertial mass has a rigorous physical meaning:

$$m = \int dV\,[t_M^{00} + t_g^{00}]. \tag{27.4}$$

It follows from this definition that the inertial mass of a body depends not only on its internal characteristics but also on the square of the velocity of this body. It is therefore to be expected that the passive gravitational mass of an extended body will also depend not only on its internal structure but on the square of its velocity. Thus, in metric theories of gravitational possessing conservation laws of matter and gravitational field taken together comparison of the passive and active gravitational masses of an extended body with its inertial mass is of undoubted interest.

How is the passive gravitational mass to be defined in this case? It is natural but somewhat formal that the tensor of passive gravitational mass of an extended body should be defined directly from the equations of motion if in the post-Newtonian approximation they can be represented in the quasi-Newtonian form

$$Ma^\alpha = m_p^{\alpha\beta}\partial_\beta \mathcal{U}, \tag{27.5}$$

where the tensor $m_p^{\alpha\beta}$ must depend only on the characteristics of the first body and the generalized Newtonian potential $\mathcal{U}$ only on the characteristics of the second body and the distance between the bodies. If these conditions are not satisfied, as is the case in the general case of an arbitrary post-Newtonian system, then the concept of a tensor of passive gravitational mass becomes pointless. It is not possible to give another more reasonable definition of the tensor of passive gravitational mass.

Thus, the solution of the question of the relation between the passive gravitational mass and the inertial mass of an extended body and of the character of the motion of its center of mass to be found in the scientific literature is incorrect, and this led us to the special study of it [2, 3].

It should, however, be noted that the initial expression we adopted in [2, 3] for the metric of Riemannian space—time is somewhat different from the corresponding expressions used by other authors. For convenience of comparison of our results with the results of other authors in the present work we have therefore chosen the initial metric of Riemannian space—time in the form (16.1). Moreover, to increase the generality of the investigation, in the present work we expand the equations of motion of the extended body in the small parameter $L/R \ll 1$ (L is a characteristic dimension of the body and R is the distance between the bodies) to higher orders than in [2, 3].

As we shall see below, however, these changes do not affect the final conclusions regarding the character of the motion of the center of mass of the extended body.

We shall consider a problem of astronomical type: we assume that the system to be studied consists of two extended bodies moving in the gravitational field they create and a distance from one another considerably greater than their linear dimensions. One of the bodies of this system we provisionally call the first body and the other the second body. We assume that these bodies consist of an ideal fluid with an energy—momentum tensor (of weight 1) having the form (16.7).

We shall also assume that the post-Newtonian formalism is applicable to this system. For this it is necessary that the maximal values of the gravitational potential U, the square of the characteristic velocity v^2, the specific pressure p/ρ_0, and the specific internal energy Π have approximately the same order of smallness ε^2, where $\varepsilon \ll 1$ is some dimensionless parameter. In this case the bodies will be located in the near zone of the gravitational radiation caused by their motion. Therefore, in the region occupied by these bodies the changes of all quantities with time will be caused primarily by the motion of matter and hence the partial derivatives of all quantities with respect to time will be small as compared to the partial derivatives with respect to the coordinates. As is known [11], any theory of gravitation in which the natural geometry for the motion of matter is a Riemannian geometry generates the metric (16.1) in the post-Newtonian approximation.

This metric in the general case contains 10 arbitrary parameters γ, β, α_1, α_2, α_3, ξ_1, ξ_2, ξ_3, ξ_4, ξ_W and three components w^α of the velocity of the reference system relative to some hypothetical universal rest system. In computing the motion of the bodies in [9, 7] it was assumed that $w^\alpha = 0$, $\xi_W = 0$. In the present work we shall use the metric of Riemannian space (16.1) without this simplification. We note that the model of extended bodies we adopt describes bodies in which the pressure is isotropic. Therefore, our calculation is applicable only to those physical situations where the magnitude of the shear stresses in extended bodies can be neglected as compared with the magnitude of the isotropic pressure. If this is not the case, then it is necessary to consider the contribution of shear stresses both in the energy—momentum tensor of matter (16.7) and in the metric (16.1).

It should also be emphasized that the calculation we propose is applicable only to those metric theories of gravitation which possess conservation laws of the energy—momentum of matter and the gravitational field taken together. For theories of gravitation not possessing these conservation laws, the calculation must be carried out specially within the framework of each such theory, and the conclusions of the present work are not applicable to them.

28. Acceleration of the Center of Mass of an Extended Body in a Weak Gravitational Field

To define the force by which the second body acts on the first we must construct the equations of hydrodynamics (the equations of motion of an element of ideal fluid) in Riemannian space—time with metric (16.1). Following Fock [6], to construct these equations we proceed from the covariant equation of the density of the energy—momentum tensor of matter